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Free vibration analysis of circularly curved multi-span Timoshenko beams by the pseudospectral method

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Abstract

The pseudospectral method is applied to the free vibration analysis of circularly curved multi-span Timoshenko beams. Each section of the beam has its own basis functions, and the continuity conditions at the intermediate supports as well as the boundary condition are treated as the constraints of the basis functions so that the number of degrees of freedom matches the number of the pseudospectral expansion coefficients. The computed natural frequencies are compared with those of existing literature, where it is shown that they are in good agreement. Numerical examples are provided for pinned-pinned, clamped-clamped and free-pinned boundary conditions for different numbers of sections and for different thickness-to-length ratios.

Keywords: Free vibration; Pseudospectral method; Phebyshev polynomials; Timoshenko beam; Multi-span beam; Curved beam

1. Introduction

A curved multi-span beam is one of the important engineering elements in mechanical and civil applications, as can be found in rail systems and bridges. The free vibration analysis of curved single-span beams based on the Timoshenko beam theory has been studied by using various methods such as the transfer matrix method [1-3], the dynamic stiffness method [4, 5], the differential quadrature method [6], and the finite element method [7-9]. The research on the free vibration of straight multi-span beams based on the Timoshenko theory also has been carried out by the Rayleigh-Ritz method [10] and the transfer matrix method [11]. However, research on the free vibration of curved multi-span Timoshenko beams has scarcely been reported. Howson and Jemah computed the natural frequencies of a circularly curved double-span Timoshenko beam using the dynamic stiffness method with the Wittrick-Williams algorithm, where they did not account for how the continuity conditions at the intermediate support were dealt with [12]. In the dynamic stiffness method the beam is discretized into a number of elements, and a frequency-dependent stiffness matrix is derived for each element. The dynamic stiffness matrix for the overall structure is assembled from the element matrices. The frequencies at which the determinant of the assembled stiffness matrix vanishes give the natural frequencies. However, it has a major drawback in that it is prone to miss the roots.

Recently, Lee applied the pseudospectral method to the free vibration analysis of straight double-span Timoshenko beams, where the continuity conditions at the intermediate support were treated as the constraints [13]. The system of equations was such that only the terms containing the natural frequency were placed at the right hand side of the equations, so that standard subroutines could be utilized to find the eigenvalues.

In the present study, the pseudospectral method is applied to the free vibration analysis of circularly curved multi-span Timoshenko beams.

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2. Pseudospectral formulations for circularly curved multi-span beams

Consider a circularly curved beam of radius R and of total angle Θ_M, which has $M-1$ roller supports at $\theta=\Theta_i$ $(i=1,\cdot\cdot,M-1)$ so that the curved beam is divided into M sections as depicted in Fig. 1. The equations of out-of-plane motion in the interval $\Theta_{i-1}<\theta<\Theta_i$ $(i=1,\cdot\cdot,M)$ are given as [12]

$$\begin{gathered}\frac{EI}{R^2}\frac{d^2\psi}{d\theta^2}-\left(\frac{GJ}{R^2}+\alpha AG\right)\psi+\frac{EI+GJ}{R^2}\frac{d\phi}{d\theta}-\frac{\alpha AG}{R}\frac{dw}{d\theta}=-\omega^2\rho I\psi\,,\\ \frac{\alpha AG}{R}\frac{d\psi}{d\theta}+\frac{\alpha AG}{R^2}\frac{d^2w}{d\theta^2}=-\omega^2\rho Aw\,,\\ -\frac{EI+GJ}{R^2}\frac{d\psi}{d\theta}+\frac{GJ}{R^2}\frac{d^2\phi}{d\theta^2}-\frac{EI}{R^2}\phi=-\omega^2\rho I_p\phi\end{gathered} \tag{1}$$

where w, ψ and ϕ are the lateral deflection, the rotation of the normal line and the torsional rotation, respectively. E and G are Young's modulus and the shear modulus, α is the shear correction factor. A, I and I_p are the cross sectional area of the beam, the second moment of area and the polar moment of area, respectively. ρ is the density, and ω is the natural frequency in radian/second. Howson and Jemah [12] introduced parameters such as

$$\lambda=\sqrt{\frac{\mathrm{AP}^2}{\mathrm{I}}},\quad \gamma=\sqrt{\frac{\rho\omega^2\mathrm{P}^2}{\mathrm{E}}} \tag{2}$$

in their derivations, which are also employed in this study.

Typical boundary conditions are

$$\begin{aligned}&\textit{clamped:}\ \psi=0,\ \phi=0,\ w=0\\ &\textit{pinned:}\ \frac{d\psi}{d\theta}=0,\ \phi=0,\ w=0\\ &\textit{free:}\ \frac{d\psi}{d\theta}+\phi=0,\ -\psi+\frac{d\phi}{d\theta}=0,\ \psi+\frac{1}{R}\frac{dw}{d\theta}=0\end{aligned} \tag{3}$$

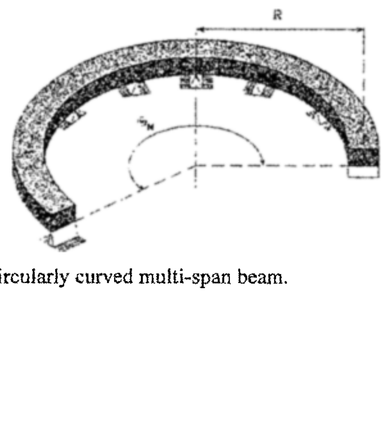

Fig. 1. Circularly curved multi-span beam.

at $\theta=0$ and $\theta=\Theta_M$. The continuity conditions at the intermediate support located at $\theta=\Theta_i$ $(i=1,2,\cdot\cdot,M-1)$ are represented by

$$\begin{aligned}&\psi\left(\theta=\Theta_i^-\right)=\psi\left(\theta=\Theta_i^+\right)\,,\\ &\frac{d\psi}{dx}\left(\theta=\Theta_i^-\right)=\frac{d\psi}{dx}\left(\theta=\Theta_i^+\right)\,,\\ &\phi\left(\theta=\Theta_i^-\right)=0\,,\\ &\phi\left(\theta=\Theta_i^+\right)=0\,,\\ &w\left(\theta=\Theta_i^-\right)=0\,,\\ &w\left(\theta=\Theta_i^+\right)=0\end{aligned} \tag{4}$$

It is convenient that each section of the curved beam between the supports is represented by a normalized local coordinate z_i

$$z_i=\frac{2\theta-\Theta_{i-1}-\Theta_i}{\Theta_i-\Theta_{i-1}}=\frac{2\theta-\Theta_{i-1}-\Theta_i}{\Delta\Theta_i}\in[-1,1]$$
$$\text{for}\ \ \Theta_{i-1}<\theta<\Theta_i\ (i=1,2,\cdot\cdot,M) \tag{5}$$

Using the relationship of (5), the governing Eq. (1) can be rewritten as

$$\begin{gathered}\frac{EI}{R^2}\left(\frac{2}{\Delta\Theta_i}\right)^2\psi''-\left(\frac{GJ}{R^2}+\alpha AG\right)\psi+\frac{EI+GJ}{R^2}\frac{2}{\Delta\Theta_i}\phi'\\ -\frac{\alpha AG}{R}\frac{2}{\Delta\Theta_i}w'=-\omega^2\rho I\psi\,,\\ \frac{\alpha AG}{R}\frac{2}{\Delta\Theta_i}\psi'+\frac{\alpha AG}{R^2}\left(\frac{2}{\Delta\Theta_i}\right)^2w''=-\omega^2\rho Aw\,,\\ -\frac{EI+GJ}{R^2}\frac{2}{\Delta\Theta_i}\psi'+\frac{GJ}{R^2}\left(\frac{2}{\Delta\Theta_i}\right)^2\phi''-\frac{EI}{R^2}\phi=-\omega^2\rho I_p\phi\,,\\ (i=1,2,\cdot\cdot,M)\end{gathered} \tag{6}$$

where ' stands for differentiation with respect to z_i. Variables $\psi(z_i)$, $\phi(z_i)$ and $w(z_i)$ are approximated by partial sums as follows:

$$\begin{gathered}\psi(z_i)\cong\tilde{\psi}(z_i)=\sum_{k=1}^{K+2}a_{ik}T_{k-1}(z_i)\,,\\ \phi(z_i)\cong\tilde{\phi}(z_i)=\sum_{k=1}^{K+2}b_{ik}T_{k-1}(z_i)\,,\\ w(z_i)\cong\tilde{w}(z_i)=\sum_{k=1}^{K+2}c_{ik}T_{k-1}(z_i)\,,\\ (i=1,2,\cdot\cdot,M)\end{gathered} \tag{7}$$

where a_{ik} , b_{ik} and c_{ik} are the expansion coefficients. K is the number of collocation points and T_{k-1} is the Chebyshev polynomial of the first kind of degree of $k-1$, respectively.

Expansions (7) are substituted into (6), and are collocated at the Gauss-Lobatto collocation points

$$\xi_j = -\cos\frac{\pi(2j-1)}{2K} \quad (j=1,2,\cdots,K) \tag{8}$$

to yield

$$\begin{gathered}
\sum_{k=1}^{K+2} a_{ik}\left[\frac{4EI}{R^2\Delta\Theta_i^2}T''_{k-1}(\xi_j)-\left(\frac{GJ}{R^2}+\alpha AG\right)T_{k-1}(\xi_j)\right]+\sum_{k=1}^{K+2} b_{ik}\frac{2(EI+GJ)}{R^2\Delta\Theta_i}T'_{k-1}(\xi_j) \\
-\sum_{k=1}^{K+2} c_{ik}\frac{2\alpha AG}{R\Delta\Theta_i}T'_{k-1}(\xi_j) = -\omega^2\rho I\sum_{k=1}^{K+2} a_{ik}T_{k-1}(\xi_j) , \\
\sum_{k=1}^{K+2} a_{ik}\frac{2\alpha AG}{R\Delta\Theta_i}T'_{k-1}(\xi_j)+\sum_{k=1}^{K+2} c_{ik}\frac{4\alpha AG}{R^2\Delta\Theta_i^2}T''_{k-1}(\xi_j) = -\omega^2\rho A\sum_{k=1}^{K+2} c_{ik}T_{k-1}(\xi_j) , \\
-\sum_{k=1}^{K+2} a_{ik}\frac{2(EI+GJ)}{R^2\Delta\Theta_i}T'_{k-1}(\xi_j)+\sum_{k=1}^{K+2} b_{ik}\left[\frac{4GJ}{R^2\Delta\Theta_i^2}T''_{k-1}(\xi_j)-\frac{EI}{R^2}T_{k-1}(\xi_j)\right] \\
= -\omega^2\rho I_p\sum_{k=1}^{K+2} b_{ik}T_{k-1}(\xi_j) , \\
(i=1,2,\cdots,M),\ (j=1,2,\cdots,K)
\end{gathered} \tag{9}$$

Eq. (9) can be rearranged in the matrix form

$$[P]\{\delta\}+[P^*]\{\delta^*\} = \omega^2\left([Q]\{\delta\}+[Q^*]\{\delta^*\}\right) \tag{10}$$

where the vectors in (10) are

$$\begin{aligned}
\{\delta\} = \{ & a_{11}\ a_{12}\cdots a_{1K}\ b_{11}\ b_{12}\cdots b_{1K}\ c_{11}\ c_{12}\cdots \\
& c_{1K}\ a_{21}\ a_{22}\cdots a_{2K}\ b_{21}\ b_{22}\cdots b_{2K}\ c_{21}\ c_{22}\cdots c_{2K}\cdots \\
& a_{M1}\ a_{M2}\cdots a_{MK}\ b_{M1}\ b_{M2}\cdots b_{MK}\ c_{M1}\ c_{M2}\cdots c_{MK}\}^T \\
\{\delta^*\} = \{ & a_{1(K+1)}\ a_{1(K+2)}\ b_{1(K+1)}\ b_{1(K+2)}\ c_{1(K+1)}\ c_{1(K+2)}\cdots \\
& a_{M(K+1)}\ a_{M(K+2)}\ b_{M(K+1)}\ b_{M(K+2)}\ c_{M(K+1)}\ c_{M(K+2)}\}^T
\end{aligned} \tag{11}$$

The total number of equations in (9) is $3MK$, whereas the total number of unknowns is $3M(K+2)$. The remaining $6M$ equations are obtained from the continuity conditions and the boundary conditions. Using the expansions of Eq. (7) the continuity conditions at the intermediate support at $\theta=\Theta_i$ are expressed by

$$\begin{gathered}
\sum_{k=1}^{K+2} a_{ik}T_{k-1}(1) = \sum_{k=1}^{K+2} a_{(i+1)k}T_{k-1}(-1) , \\
\frac{1}{\Delta\Theta_i}\sum_{k=1}^{K+2} a_{ik}\frac{dT_{k-1}(1)}{dz_i} = \frac{1}{\Delta\Theta_{i+1}}\sum_{k=1}^{K+2} a_{(i+1)k}\frac{dT_{k-1}(-1)}{dz_{i+1}} , \\
\sum_{k=1}^{K+2} b_{ik}T_{k-1}(1) = 0 , \\
\sum_{k=1}^{K+2} b_{(i+1)k}T_{k-1}(-1) = 0 , \\
\sum_{k=1}^{K+2} c_{ik}T_{k-1}(1) = 0 , \\
\sum_{k=1}^{K+2} c_{(i+1)k}T_{k-1}(-1) = 0
\end{gathered} \tag{12}$$

The boundary conditions are

$$\begin{aligned}
\textit{clamped}: \quad & \sum_{k=1}^{K+2} a_{1k}T_{k-1}(-1)=0,\ \sum_{k=1}^{K+2} b_{1k}T_{k-1}(-1)=0, \\
& \sum_{k=1}^{K+2} c_{1k}T_{k-1}(-1)=0 \\
\textit{pinned}: \quad & \sum_{k=1}^{K+2} a_{1k}\frac{dT_{k-1}(-1)}{dz_1}=0,\ \sum_{k=1}^{K+2} b_{1k}T_{k-1}(-1)=0, \\
& \sum_{k=1}^{K+1} c_{1k}T_{k-1}(-1)=0 \\
\textit{free}: \quad & \left[\begin{aligned}
& \sum_{k=1}^{K+2}\left\{\frac{2a_{1k}}{\Delta\Theta_1}\frac{dT_{k-1}(-1)}{dz_1}+b_{1k}T_{k-1}(-1)\right\}=0, \\
& \sum_{k=1}^{K+2}\left\{-a_{1k}T_{k-1}(-1)+\frac{2b_{1k}}{\Delta\Theta_1}\frac{dT_{k-1}(-1)}{dz_1}\right\}=0, \\
& \sum_{k=1}^{K+2}\left\{a_{1k}T_{k-1}(-1)+\frac{2c_{1k}}{R\Delta\Theta_1}\frac{dT_{k-1}(-1)}{dz_1}\right\}=0
\end{aligned}\right.
\end{aligned} \tag{13}$$

at $\theta=0$, and

$$\begin{aligned}
\textit{clamped}: \quad & \sum_{k=1}^{K+2} a_{Mk}T_{k-1}(1)=0,\ \sum_{k=1}^{K+2} b_{Mk}T_{k-1}(1)=0, \\
& \sum_{k=1}^{K+2} c_{Mk}T_{k-1}(1)=0 \\
\textit{pinned}: \quad & \sum_{k=1}^{K+2} a_{Mk}\frac{dT_{k-1}(1)}{dz_M}=0,\ \sum_{k=1}^{K+2} b_{Mk}T_{k-1}(1)=0, \\
& \sum_{k=1}^{K+1} c_{Mk}T_{k-1}(1)=0 \\
\textit{free}: \quad & \left[\begin{aligned}
& \sum_{k=1}^{K+2}\left\{\frac{2a_{Mk}}{\Delta\Theta_M}\frac{dT_{k-1}(1)}{dz_M}+b_{Mk}T_{k-1}(1)\right\}=0, \\
& \sum_{k=1}^{K+2}\left\{-a_{Mk}T_{k-1}(1)+\frac{2b_{Mk}}{\Delta\Theta_M}\frac{dT_{k-1}(1)}{dz_M}\right\}=0, \\
& \sum_{k=1}^{K+2}\left\{a_{Mk}T_{k-1}(1)+\frac{2c_{Mk}}{R\Delta\Theta_M}\frac{dT_{k-1}(1)}{dz_M}\right\}=0
\end{aligned}\right.
\end{aligned} \tag{14}$$

at $\theta = \Theta_M$. The continuity conditions (12) and the boundary condition set, one at $\theta = 0$ and another at $\theta = \Theta_M$, can be rearranged in the matrix form

$$[W]\{\delta\} + [Y]\{\delta^*\} = \{0\} \tag{15}$$

where $\{0\}$ is a zero vector. Because $\{\delta^*\}$ in Eq. (15) can be expressed by

$$\{\delta^*\} = -[Y]^{-1}[W]\{\delta\}, \tag{16}$$

Eq. (10) is rearranged into

$$\left([P] - [P^*][Y]^{-1}[W]\right)\{\delta\} = \omega^2\left([Q] - [Q^*][Y]^{-1}[W]\right)\{\delta\} \tag{17}$$

The solution of (17) yields an estimate for the natural frequencies and the corresponding mode shapes.

3. Numerical examples

Howson and Jemah [12] scaled the torsional rigidity of the curved double-span beam to $GJ/EI = 10^3$ so that only the natural frequencies corresponding to predominantly flexural modes appeared. The comparison of the natural frequencies computed by Howson and Jemah [12] to those of the present study is given in Table 1, where it is shown that they are in

Table 1. Comparison of frequency parameter $\gamma\lambda$ of circularly curved double-span Timoshenko beam ($\Theta_2 = \pi$, $\lambda = 23.39$, square cross-section, pinned-pinned boundary condition, ν=0.3, α=0.83, $GJ/EI = 10^3$, K=20)

Mode	1	2	3	4	5	6	7	8	16
Howson and Jemah	2.967	5.394	14.24	17.89	31.29	35.57	52.43	56.82	158.8
Present Study	2.965	5.390	14.23	17.87	31.26	35.54	52.39	56.77	158.6

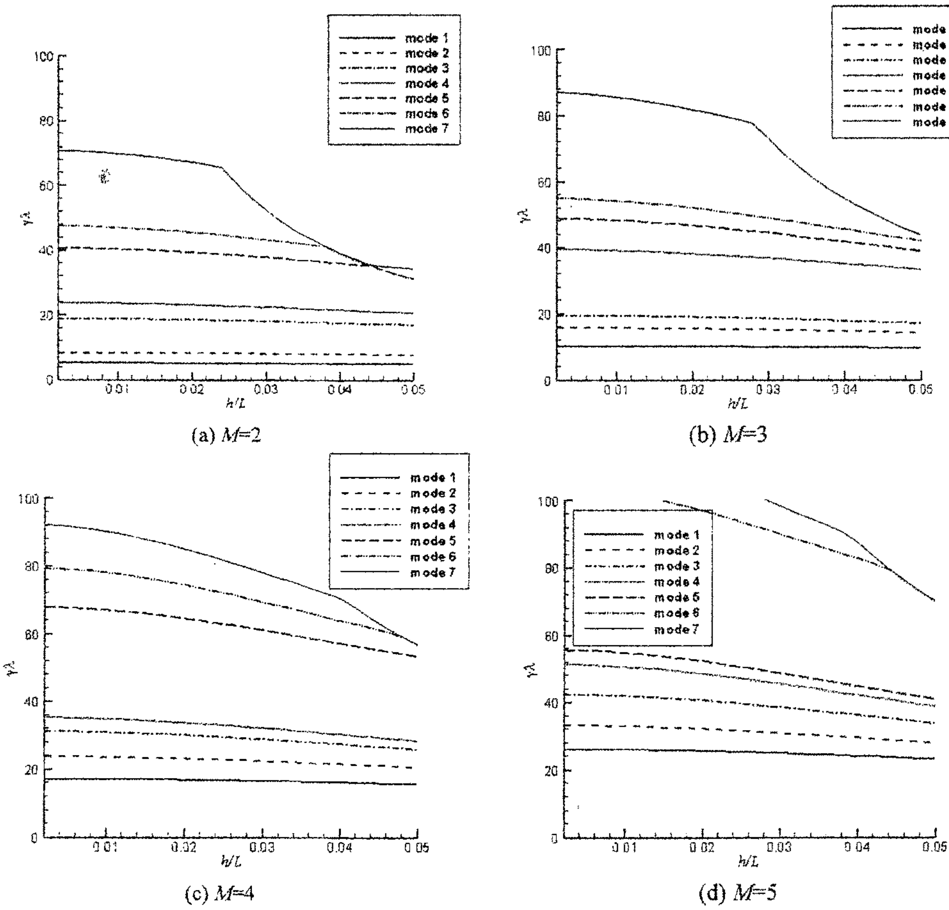

Fig. 2. Frequency parameter $\gamma\lambda$ of circularly curved multi-span Timoshenko beams ($\Theta_M = \pi$, evenly spaced supports, square cross-section, clamped-clamped boundary condition, ν=0.3, α=5/6, K=20).

good agreement. The numbers given in Table 1 and Figs. 2~4 are the frequency parameter $\gamma\lambda$ defined as

$$\gamma\lambda = \sqrt{\rho A R^4 \omega^2 / EI} \tag{18}$$

The natural frequencies of circularly curved multi-span Timoshenko beams with pinned-pinned, clamped-clamped, and pinned-free boundary conditions were computed without suppressing the torsional modes for square cross sections and for $K = 20$. The natural frequencies were calculated for different thickness-to-length ratios ranging from $h/L = 0.002$ to $h/L = 0.05$, where h and $L = R\Theta_M$ are the thickness and the length of the beam, respectively, and the computed frequency parameter $\gamma\lambda$ for the lowest seven natural frequencies are given in Figs. 2~4. It is readily seen from Figs. 2~4 that the frequency parameter $\gamma\lambda$ increased as M increased. For the low modes the frequency parameter $\gamma\lambda$ remained fairly constant as the thickness-to-length ratio h/L increased regardless of the boundary condition. On the other hand, for some higher modes, for example, the seventh mode of clamped-clamped boundary condition showed an abrupt decrease of $\gamma\lambda$ as h/L increased, which indicated that the predictions of natural frequencies of the curved multi-span beams based on the Bernoulli-Euler theory might be erroneous for higher modes and for larger thickness-to-length ratios.

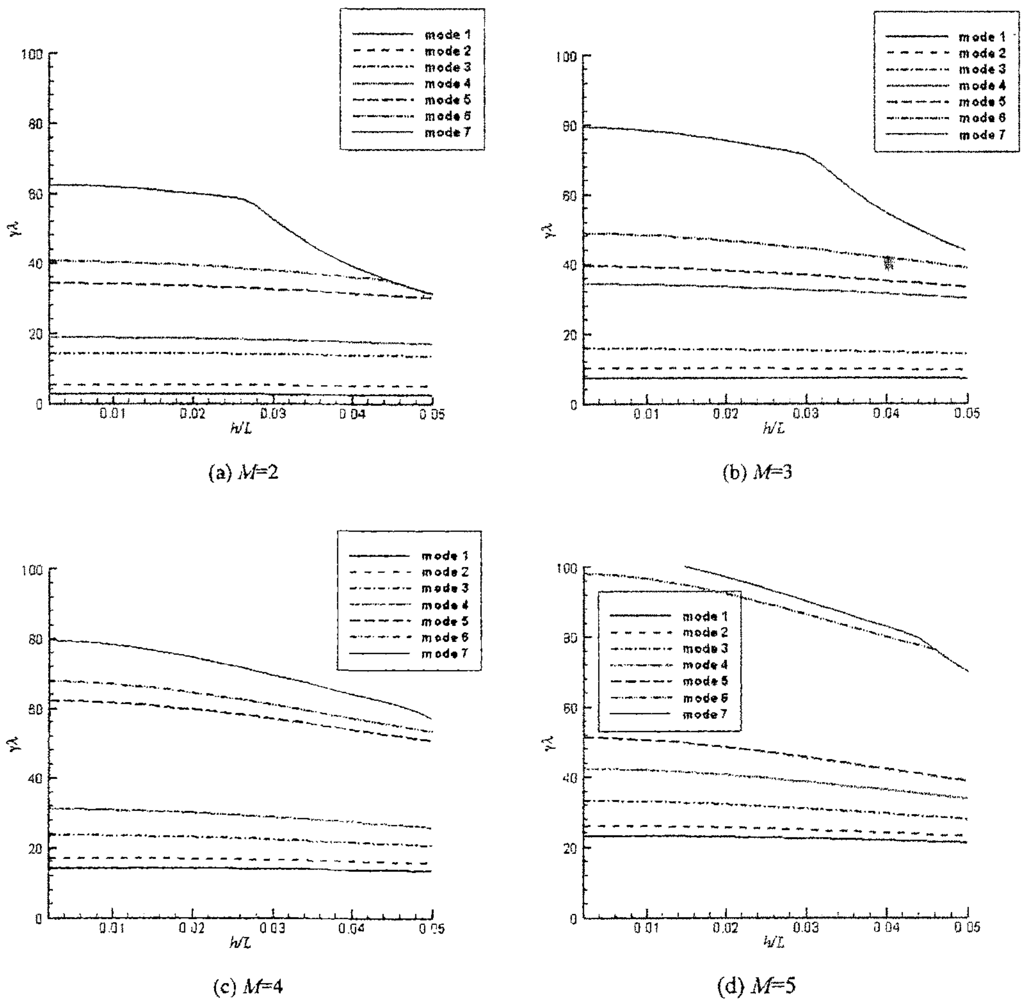

Fig. 3. Frequency parameter $\gamma\lambda$ of circularly curved multi-span Timoshenko beams ($\Theta_M = \pi$, evenly spaced supports, square cross-section, pinned-pinned boundary condition, ν=0.3, α=5/6, K=20).

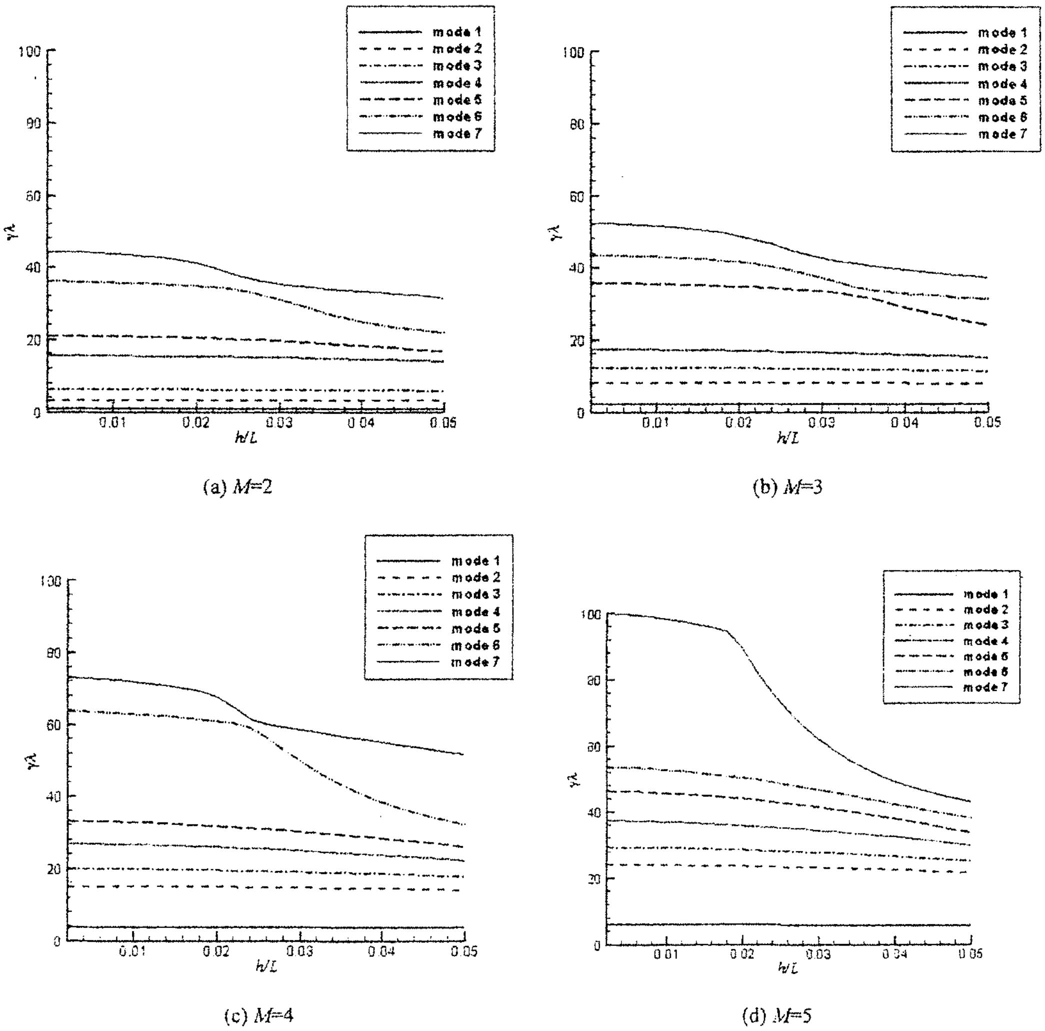

Fig. 4. Frequency parameter $\gamma\lambda$ of circularly curved multi-span Timoshenko beams ($\Theta_M = \pi$, evenly spaced supports, square cross-section, pinned-free boundary condition, ν=0.3, α=5/6, K=20).

4. Conclusions

The pseudospectral method is applied to the free vibration analyses of circularly curved multi-span Timoshenko beams. The pseudospectral method uses simple series expansions such as the Chebyshev polynomials. The formulation as well as coding for computation is straightforward because the pseudospectral method undergoes a simple collocation process instead of integration. Basis functions are assumed for each section of the curved multi-span beam. The continuity conditions at the intermediate supports and the boundary condition are considered as the side constraints, and the set of algebraic equations is condensed so that the number of degrees of freedom of the problem matches the number of the pseudospectral expansion coefficients. Numerical examples are provided for various thickness-to-length ratios and for different numbers of sections.

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